Càlcul vectorial

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Operadors vectorials

1.Coordenades cartesianes

Siguin:

$f(x,y,z)$ i $\vec{f}(\vec{r}) = f_x(\vec{r})\hat{\imath} + f_y(\vec{r})\hat{\jmath} + f_z(\vec{r})\hat{k}$

- **Gradient**:

$$\vec{\nabla} \cdot f = \left(\frac{\partial}{\partial x}f, \frac{\partial}{\partial y}f, \frac{\partial}{\partial z}f\right)$$

- **Divergència**:

$$\vec{\nabla} \cdot \vec{f} = \frac{\partial}{\partial x}f + \frac{\partial}{\partial y}f + \frac{\partial}{\partial z}f$$

- **Rotacional**:

$$\vec{\nabla} \times \vec{f} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_x & f_y & f_z \end{vmatrix}$$

- **Laplacià**:

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

2.Coordenades Polars

Siguin:

$$(x,y) \to (\rho,\theta); \quad f(\rho,\phi); \quad \vec{f} = f_\rho\hat{\rho} + f_\phi\hat{\phi}$$

- **Gradient**:

$$\vec{\nabla} \cdot f = \frac{\partial f}{\partial \rho}\hat{\rho} + \frac{\partial f}{\partial \phi}\hat{\phi}$$

- **Divergència**:

$$\vec{\nabla} \cdot \vec{f} = \frac{1}{\rho}\frac{\partial f_\rho}{\partial \rho} + \frac{\partial f_\phi}{\partial \phi}$$

- **Laplacià**:

$$\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2}$$

3.Coordenades cilíndriques

$$f(x,y,z) \to f(\rho,\phi,z); \qquad \vec{f}(\vec{r}) = (f_\rho, f_\phi, f_z)$$

- **Gradient**:

$$\vec{\nabla} \cdot f = \frac{\partial f}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\hat{\phi} + \frac{\partial f}{\partial z}\hat{z}$$

- **Divergència**:

$$\vec{\nabla} \cdot \vec{f} = \frac{1}{\rho}\frac{\partial(\rho \cdot f_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial f_\phi}{\partial \phi} + \frac{\partial f_z}{\partial z}$$

- **Rotacional**:

$$\vec{\nabla} \times \vec{f} = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \partial/\partial\rho & \partial/\partial\phi & \partial/\partial z \\ f_\rho & \rho f_\phi & f_z \end{vmatrix}$$

- **Laplacià**:

$$\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z}$$

4.Coordenades esfèriques

$$f(x,y,z) \to f(\rho,\theta,\phi); \qquad \vec{f}(\vec{r}) = (f_\rho, f_\theta, f_\phi)$$

- **Gradient**:

$$\vec{\nabla} \cdot f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\phi}$$

- **Divergència**:

$$\vec{\nabla}\cdot\vec{f} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 f_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\cdot f_\theta) + \frac{1}{r\sin\theta}\frac{\partial f_\phi}{\partial\phi}$$

- **Rotacional**:

$$\vec{\nabla} \times \vec{f} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ f_r & rf_\theta & r\sin\theta f_\phi \end{vmatrix}$$

- **Laplacià**:

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) +$$

$$+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

CORBES I SUPERFÍCIES EN $\mathbb{R}^3$

Tenim $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$

- Equació d'una recta tangent a una corba en un punt $A(x_0, y_0, z_0)$:

$$\frac{x - x_0}{\mathrm{d}x(t)/\mathrm{d}t} = \frac{y - y_0}{\mathrm{d}y(t)/\mathrm{d}t} = \frac{z - z_0}{\mathrm{d}z(t)/\mathrm{d}t}$$

- Equació del pla normal a la corba en $A \to \vec{r} - \vec{r}_0 \perp \mathrm{d}\vec{r}/\mathrm{d}t$:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t}(x - x_0) + \frac{\mathrm{d}y(t)}{\mathrm{d}t}(y - y_0) + \frac{\mathrm{d}z(t)}{\mathrm{d}t}(z - z_0) = 0$$

Parametritzar en funció de la longitud d'arc: $\vec{r}(t) = (x(t), y(t), z(t)) = r(x(z), y(z), z)$

- $\mathrm{d}s = \left|\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right| \mathrm{d}t$
- $\frac{\mathrm{d}\vec{r}}{\mathrm{d}s} \equiv \hat{\sigma}$; $\quad \lim_{\Delta S \to 0} \left|\frac{\Delta\phi}{\Delta s}\right| = K$; $\quad \frac{\mathrm{d}\hat{\sigma}}{\mathrm{d}S} = K\hat{n}$
- $R = \frac{1}{K}$; $\to \frac{1}{R^2} = \left|\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}s}\right|^2$
- $\hat{b} = \hat{\sigma} \times \hat{n}$; $\quad \frac{\mathrm{d}\hat{b}}{\mathrm{d}s} = \hat{\sigma} \times \frac{\mathrm{d}\hat{n}}{\mathrm{d}s}$; $\quad \frac{\mathrm{d}\hat{b}}{\mathrm{d}s} = \tau\hat{n} = \frac{1}{T}\hat{n}$
- $\tau = \frac{1}{T} = \left|-R^2 \cdot \frac{\mathrm{d}\vec{r}}{\mathrm{d}s}\left(\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}s^2} \times \frac{\mathrm{d}^3\vec{r}}{\mathrm{d}s^3}\right)\right|$

Fómules de Serret-Frenet: $\hat{\sigma}, \hat{b}$ i $\hat{n}$ formen un sistema de referència mòbil ortogonal, on

- $\hat{\sigma} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}s}$; $\quad \hat{n} = \frac{1}{z}\frac{\mathrm{d}\hat{b}}{\mathrm{d}s} = \frac{1}{K}\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}s^2}$; $\quad \hat{b} = \frac{1}{k}\left(\frac{\mathrm{d}\vec{r}}{\mathrm{d}s} \times \frac{\mathrm{d}^2\vec{r}}{\mathrm{d}s^2}\right)$

En funció d'un paràmetre t qualsevol:

- $\hat{\sigma}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$
- $\hat{b}(t) = \frac{\vec{r}'(t) \times \vec{r}''(t)}{|\vec{r}'(t) \times \vec{r}''(t)|}$
- $\hat{n}(t) = \frac{(\vec{r}'(t) \times \vec{r}''(t)) \times \vec{r}'(t)}{|(\vec{r}'(t) \times \vec{r}''(t)) \times \vec{r}'(t)|}$
- $\frac{1}{R} = \frac{\left|\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \times \frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2}\right|}{\left|\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right|^3}$
- $\frac{1}{T} = \left|-\frac{\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \cdot \left(\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} \times \frac{\mathrm{d}^3\vec{r}}{\mathrm{d}t^3}\right)}{\left|\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \times \frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2}\right|^2}\right|$

Diferencials

Polars

- Posició: $\mathrm{d}\vec{r} = \mathrm{d}\rho\hat{\rho} + \rho\mathrm{d}\phi\hat{\phi}$
- Longitud: $\mathrm{d}s^2 = \mathrm{d}\rho^2 + \rho^2\mathrm{d}\phi^2$
- Superfície $\mathrm{d}S = \rho\mathrm{d}\rho\phi$

Cilíndriques

- Posició: $\mathrm{d}\vec{r} = \mathrm{d}\rho\hat{\rho} + \rho\mathrm{d}\phi\hat{\phi} + \mathrm{d}z\hat{z}$
- Longitud: $\mathrm{d}s^2 = \mathrm{d}\rho^2 + \rho\mathrm{d}\phi^2 + \mathrm{d}z^2$
- Superfície $\mathrm{d}S = \rho\mathrm{d}\phi\mathrm{d}z$ (per $\rho =$cte.)
- Volum $\mathrm{d}V = \rho\mathrm{d}\rho\mathrm{d}\phi\mathrm{d}z$

Esfèriques

- Posició: $\mathrm{d}\vec{r} = \mathrm{d}r\hat{r} + r\mathrm{d}\theta\hat{\theta} + r\sin\theta\mathrm{d}\phi\hat{\phi}$
- Longitud: $\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2$
- Superfície $\mathrm{d}S = r^2\sin\theta\mathrm{d}\theta\mathrm{d}\phi$ per a $r =$cte.
- Volum $\mathrm{d}V = r^2\sin\theta\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi$