Fuerzas Centrales

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3. FUERZAS CENTRALES

$$\vec{F}(\vec{r}) = F(r)\hat{r}$$

Problema de 2 cuerpos: Si $\vec{F}_{12} = -\vec{F}_{21}$ i $\frac{\vec{F}_1}{m_1} = \frac{\vec{F}_2}{m_2}$

$$\begin{cases} m_1\ddot{\vec{r}}_1 = \vec{F}_{21} + \vec{F}_1 \\ m_2\ddot{\vec{r}}_2 = \vec{F}_{12} + \vec{F}_2 \end{cases} \mapsto \begin{cases} M\ddot{\vec{R}} = \vec{F} \\ \mu\ddot{\vec{r}} = \vec{F}_{21} \end{cases}, \text{ donde :}$$

$$M = m_1 + m_2, \quad \vec{F} = \vec{F}_1 + \vec{F}_2, \quad \vec{r} = \vec{r}_1 - \vec{r}_2,$$
$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}, \quad \mu = \frac{m_1 m_2}{m_1 + m_2}$$

Dinámica en $\mathbb{R}^3$

1. $\vec{p} = m\vec{v}; \quad \vec{I}_{1,2} = \int_{t_1}^{t_2} \vec{F}(\vec{r}, \dot{\vec{r}}, t)\mathrm{d}t \ \rightarrow \ \vec{F} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t}$

2. $\left.\begin{array}{l} \vec{L} = \vec{r} \times \vec{p} \\ \vec{N} = \vec{r} \times \vec{F} \end{array}\right\} \mapsto \frac{\mathrm{d}\vec{L}}{\mathrm{d}t} = \vec{N} \ \rightarrow \Delta\vec{L} = \int_{t_1}^{t_2} \vec{N}\mathrm{d}t$

3. $T = \frac{1}{2}m\vec{v}^{\,2}; \ W = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}(\vec{r}, \dot{\vec{r}}, t) \cdot \mathrm{d}\vec{r}; \ \frac{\mathrm{d}T}{\mathrm{d}t} = \vec{F} \cdot \vec{v}$

4. Sistemas conservativos: $\vec{F} = \vec{F}(r) \quad \vec{\nabla} \times \vec{F} = \vec{0}$

$$U(\vec{r}) = -\int_{\vec{r}_R}^{\vec{r}} \vec{F}(\vec{r}\,') \cdot \mathrm{d}\vec{r}\,' \quad \rightarrow \quad \vec{F}(\vec{r}) = -\vec{\nabla}U(\vec{r})$$
$$E = \frac{1}{2}m\vec{v}^{\,2} + U(\vec{r}) = \mathcal{C}$$

Propiedades de los campos centrales

$$\vec{F}(\vec{r}) = F(r)\hat{r} = \frac{x}{r}F(r)\hat{\imath} + \frac{y}{r}F(r)\hat{\jmath} + \frac{z}{r}F(r)\hat{k}$$
$$\vec{L} = \vec{\mathcal{C}}, \quad \Delta E = 0, \quad \vec{\nabla} \times F(r)\hat{r} = \vec{0}$$
$$U(r) = -\int_{r_s}^{r} F(r')\mathrm{d}r' \quad \vec{F} = -\vec{\nabla}U(\vec{r}); \quad F(r) = -\frac{\mathrm{d}U}{\mathrm{d}r}$$
$$E = \frac{1}{2}m\vec{v}^{\,2} + U(r)$$

Métodos de resolución

$$x = r\cos\varphi, \ y = r\sin\varphi, \ r = \sqrt{x^2 + y^2}, \ \varphi = \arctan\tfrac{y}{x}$$

$$\hat{r} = \cos\varphi\hat{\imath} + \sin\varphi\hat{\jmath}, \qquad \hat{\varphi} = -\sin\varphi\hat{\imath} + \cos\varphi\hat{\jmath}$$
$$\dot{\hat{r}} = \dot{\varphi}\hat{\varphi}, \qquad \dot{\hat{\varphi}} = -\dot{\varphi}\hat{r}$$

1. Aplicación de la segunda ley de Newton

$$\vec{r} = r\hat{r}$$
$$\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\varphi}\hat{\varphi}$$
$$\ddot{\vec{r}} = (\ddot{r} - r\dot{\varphi}^2)\hat{r} + (2\dot{r}\dot{\varphi} + r\ddot{\varphi})\hat{\varphi}$$

$$m\ddot{\vec{r}} = \vec{F}(\vec{r}) = F(r)\hat{r} + 0 \cdot \hat{\varphi} \quad \rightarrow \begin{cases} m(\ddot{r} - r\dot{\varphi}^2) = F(r) \\ 2\dot{r}\dot{\varphi} + r\ddot{\varphi} = 0 \end{cases}$$

$$L = mr^2\dot{\varphi} = \mathcal{C}$$

$$\begin{cases} m\ddot{r} = F(r) + \dfrac{L}{mr^3} \equiv F'(r) \\ \dot{\varphi} = \dfrac{L}{mr^2} \end{cases}$$

2. Teoremas de conservación

$$\begin{cases} L = mr^2\dot{\varphi} \\ E = \frac{1}{2}m\vec{v}^2 + U(\vec{r}) \end{cases} \rightarrow \begin{cases} L = mr^2\dot{\varphi} \\ E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\varphi}^2 + U(r) \end{cases}$$

$$\dot{\varphi}^2 = \tfrac{L^2}{m^2r^4}$$

$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{L^2}{mr^2} + U(r) = \frac{1}{2}m\dot{r}^2 + U'(r)$$

- $\displaystyle\int_{r_0}^{r} \frac{\mathrm{d}r'}{\sqrt{\frac{2}{m}[E - U'(r')]}} = \int_0^t \mathrm{d}t' \ \mapsto \ r(t) = \Phi^{-1}\left(\sqrt{\frac{2}{m}}t\right)$
- $\displaystyle\varphi(t) = \int_0^t \frac{L}{mr^2(t')}\mathrm{d}t'$

3. Análisis semicuantitativo de $U'(r)$: $E \geq U'(r)$

- *Puntos de retorno*: $\left(\frac{\mathrm{d}U'}{\mathrm{d}r}\right)_{r_R} \neq 0; \ E = U'(r_R)$
- *Puntos de equilibrio*:

$$\left(\frac{\mathrm{d}U'}{\mathrm{d}r}\right)_{r_E} = 0 \ \rightarrow \begin{cases} \left(\dfrac{\mathrm{d}^2U'}{\mathrm{d}r}\right)_{r_E} > 0 \ \rightarrow \text{ estable} \\ \left(\dfrac{\mathrm{d}^2U'}{\mathrm{d}r}\right)_{r_E} < 0 \ \rightarrow \text{ inestable} \end{cases}$$

4. Ecuación de la trayectoria

$$r = r(t)\varphi = \varphi(t)\} \ \rightarrow r = r(\varphi) \qquad u(\varphi) \equiv \frac{1}{r(\varphi)}$$

$$\frac{\mathrm{d}^2u}{\mathrm{d}\varphi^2} + \frac{m}{L^2u^2}F\left(\frac{1}{u}\right) + u = 0$$

Fuerza del tipo $F(r) = \frac{k}{r^2}$

Estudio semicuantitativo usando $U'(r)$

$$U'(r) = U(r) + \frac{L^2}{2mr^2} \mapsto U'(r) = \frac{k}{r} + \frac{L^2}{2mr^2}$$

1. $\boxed{k > 0}$ fuerza repulsiva; No extremos, no r_E, sólo 1 r_R.

$$\frac{1}{r_1} = -\frac{mK}{L^2} + \sqrt{\left(\frac{mk}{L^2}\right)^2 + \frac{2mE}{L^2}} > 0 \rightarrow \text{Órbita ilimitada abierta}$$

2. $\boxed{k < 0}$ $E \geq U_0$

$$r_0 = -\frac{L^2}{mk} \qquad U_0 = U'(r_0) = -\frac{mk^2}{2L^2}$$

- $E = U_0$

$$\text{M.R.U} \rightarrow \begin{cases} r_0 = -\dfrac{L^2}{mk} \\ \dot{\varphi}_0 = \dfrac{L}{mr_0^2} \end{cases}$$

- $U_0 < E < 0$

$$\frac{1}{r_{1,2}} = -\frac{mk}{L^2} \pm \sqrt{\left(\frac{mk}{L^2}\right)^2 - \frac{2m|E|}{L^2}}$$

- $E \geq 0$

$$\frac{1}{r_1} = -\frac{mk}{L^2} + \sqrt{\left(\frac{mk}{L^2}\right)^2 + \frac{2mE}{L^2}}$$

Ecuación de la trayectoria

$$u(\varphi) = A\cos\varphi - \frac{mk}{L^2}$$

Determinar A $\rightarrow \varphi = 0$ ó $\varphi = \pi$:

$$A = \sqrt{\left(\frac{mk}{L^2}\right)^2 + \frac{2mE}{L^2}}$$

$$\boxed{r(\varphi) = \frac{1}{A\cos\varphi + B}} \qquad B = -\frac{mk}{L^2}$$

1. ELIPSE: $E < 0,\ k < 0\ \ \varepsilon < 1$

$$r(\varphi) = \frac{a(1-\varepsilon^2)}{1+\varepsilon\cos\varphi} = \frac{1}{B + A\cos\varphi}$$

2. HIPÉRBOLA: $E > 0$ si $k > 0$ rama - , si $k < 0$ rama +, $\varepsilon > 1$

$$r(\varphi) = \frac{a(\varepsilon^2-1)}{\pm 1+\varepsilon\cos\varphi} = \frac{1}{B + A\cos\varphi}$$

$$\varepsilon = \frac{A}{|B|} \qquad a = \left|\frac{B}{A^2 - B^2}\right|$$

3. PARÁBOLA: $E = 0;\ k < 0;\ \ \varepsilon = 1$

$$r(\varphi) = \frac{a}{1+\cos\varphi}$$

$$\varepsilon = \sqrt{1 + \frac{2EL^2}{mk^2}} \qquad a = \left|\frac{k}{2E}\right|$$

Problema de Kepler

1. Los planetas siguen órbitas elítpticas con el sol en uno de sus focos.
2. La velocidad aerolar es constante. $\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{L}{2m}$
3. $T^2 \propto a^3$ $\qquad T = \frac{4\pi^2}{GM}a^3$