Formulari de Mecànica Quàntica
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I. Formalisme i Postulats

Importants:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k \qquad [\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$$

$$|+\rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad |-\rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad |+\rangle_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \quad |-\rangle_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} \quad |+\rangle_\alpha = \begin{pmatrix} \cos\frac{\alpha}{2} \\ \sin\frac{\alpha}{2} \end{pmatrix} \quad |-\rangle_\alpha = \begin{pmatrix} -\sin\frac{\alpha}{2} \\ \cos\frac{\alpha}{2} \end{pmatrix}$$

$$\vec{S} = \frac{\hbar}{2}\vec{\sigma}$$

$$\mathrm{e}^{i\alpha\hat{n}\cdot\vec{\sigma}} = \cos\alpha\mathbb{I} + i\sin\alpha(\hat{n}\cdot\vec{\sigma})$$

Bras, kets i operadors:

$$\langle u|\, v\rangle = \sum_{i=1}^{n} u_i^* v_i \qquad \langle\psi|\,\varphi\rangle = \int_{\mathbb{R}^3} \mathrm{d}^3x\ \psi^*(\vec{x})\varphi(\vec{x}) \qquad |\psi\rangle \ \rightarrow\ \langle\psi|\,\psi\rangle = 1 \qquad A^\dagger = \left(A^{\mathrm{T}}\right)^*$$

$$E_1 = |e_1\rangle\langle e_1| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad E_2 = |e_2\rangle\langle e_2| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad E_i = E_i^\dagger = E_i^2 \qquad E_iE_j = \delta_{ij}E_i \qquad E_1 + E_2 = \mathbb{I}$$

$$\mathbb{I} = \sum_i E_i = \sum_i |e_i\rangle\langle e_i|$$

$$A|a_n\rangle = a_n|a_n\rangle \quad A = \sum_n a_n|a_n\rangle\langle a_n| = \sum_n a_nE(a_n) \quad f(A) = \sum_i f(a_i)E(a_i)$$

$$A = A^\dagger \quad \rightarrow \quad \lambda_i \in \mathbb{R} \quad \rightarrow \quad \langle\lambda_i|\,\lambda_j\rangle = \delta_{ij} \qquad A|v_i\rangle = \lambda_i|v_i\rangle$$

$$\langle A\rangle_{|v\rangle} = \langle v|\,A\,|v\rangle = \sum_i a_i|\,\langle a_i|\,v\rangle\,|^2 \geq 0$$

$$U = \mathrm{e}^{iA} = \sum_j \mathrm{e}^{ia_j}|a_j\rangle\langle a_j| \qquad UU^\dagger = \mathbb{I} \qquad \lambda_i = \mathrm{e}^{i\theta} \qquad U|u_i\rangle = |v_i\rangle \quad \langle u_i|\,u_j\rangle = \delta_{ij} = \langle v_i|\,v_j\rangle$$

$$[A,B] = AB - BA = -[B,A] \qquad [A,BC] = B[A,C] + [A,B]C \quad [A,[A,B]] = 0 \quad \rightarrow \quad [f(A),B] = [A,B]f'(A)$$

$$A = A^\dagger,\ B = B^\dagger \text{ i } [A,B] = 0 \quad \Rightarrow \quad \exists C = C^\dagger \mid A = f(C) \text{ i } B = f(C)$$

$$X = X^\dagger = \int_{-\infty}^{\infty} \mathrm{d}x\ x\,|x\rangle\langle x| \qquad \psi(x) = \langle x|\,\psi\rangle \qquad \langle x|\,y\rangle = \delta(x-y) \qquad P = -i\hbar\frac{\mathrm{d}}{\mathrm{d}x} = P^\dagger = \int_{-\infty}^{\infty} \mathrm{d}p\ p\,|p\rangle\langle p|$$

Estats i Observables Producte:

$$|\psi\rangle = |\varphi\rangle_A \otimes |\xi\rangle_B + |\phi\rangle_A \otimes |\eta\rangle_B = |\varphi\rangle_A|\xi\rangle_B + |\phi\rangle_A|\eta\rangle_B = |\varphi\xi\rangle + |\phi\eta\rangle \ \in \mathscr{H}_A \otimes \mathscr{H}_B$$

$$X^{(A)} \otimes Y^{(B)} \ \Rightarrow\ X^{(A)} \otimes Y^{(B)}|\psi\rangle = X|\varphi\rangle_A \otimes Y|\xi\rangle_B + X|\phi\rangle_A \otimes Y|\eta\rangle_B$$

Postulat de la mesura:

$$|\psi\rangle \quad \rightarrow \quad A|a_n\rangle = a_n|a_n\rangle \qquad P_{A,|\psi\rangle} = |\langle a_i|\,\psi\rangle|^2$$

$$\langle A\rangle_{|\psi\rangle} = \langle\psi|\,A\,|\psi\rangle = \sum_i a_iP(a_i) = \sum_i a_i|\,\langle a_i|\,\psi\rangle\,|^2$$

$$\Delta_{|\psi\rangle}A = \left|\left\langle\psi\right|(A - \langle A\rangle_{|\psi\rangle})^2\left|\psi\right\rangle\right|^{1/2} = \left|\langle\psi|\,A^2\,|\psi\rangle - \langle\psi|\,A\,|\psi\rangle\right|^{1/2} = \left|(A - \langle A\rangle_{|\psi\rangle})\,|\psi\rangle\right|$$

$$|\psi\rangle = |a_i\rangle \quad \rightarrow \quad P_{A,|a_i\rangle}(a_j) = |\langle a_i|\,a_j\rangle| = \delta_{ij}$$

Col·lapse: $A|\psi\rangle_i = |\phi\rangle \quad \rightarrow \quad |\psi\rangle_f = |\phi\rangle$

Principi d'incertesa:

$$[X_i, X_j] = [P_i, P_j] = 0 \qquad [X_i, P_j] = i\hbar\delta_{ij}\mathbb{I} \qquad \Delta A\cdot\Delta B \geq \frac{1}{2}|\langle[A,B]\rangle| \qquad \Delta X\cdot\Delta P_x \geq \frac{\hbar}{2}$$

$$\Delta X\Delta P_x = \frac{\hbar}{2} \qquad \rightarrow \qquad \psi(x) = C\exp\left(i\frac{X\langle P\rangle}{\hbar}\right)\exp\left(-i(\Delta P)^2\frac{(X - \langle X\rangle)^2}{\hbar^2}\right)$$

Equació d'Schrödinger. Evolució temporal:

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle = H(t)|\psi(t)\rangle \qquad \langle x|\psi(t)\rangle \quad \to \quad i\hbar\frac{\partial}{\partial t}\psi(t,x) = \left(-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}+V(x)\right)\psi(t,x)$$

$$|\psi(t)\rangle = U(t,t_0)|\psi(t_0)\rangle \quad \to \quad i\hbar\frac{\mathrm{d}}{\mathrm{d}t}U(t,t_0)H(t)U(t,t_0) \quad \to \quad U(t,t_0) = \mathrm{e}^{-\frac{i}{\hbar}(t-t_0)H} \text{ per a } H\neq H(t)$$

$$H|E\rangle = E|E\rangle \qquad |\psi(t_0)\rangle = |E\rangle \quad \to \quad |\psi(t)\rangle = \mathrm{e}^{-i\frac{t-t_0}{\hbar}E}|E\rangle$$

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\langle\psi(t)|A|\psi(t)\rangle = \langle\psi(t)|[H,A]|\psi(t)\rangle$$

$$\frac{\partial}{\partial t}\rho(t,\vec{x})+\vec{\nabla}\cdot\vec{\jmath}(t,\vec{x}) = 0 \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left\langle\vec{X}\right\rangle = \frac{1}{m}\left\langle\vec{P}\right\rangle \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left\langle\vec{P}\right\rangle = -\left\langle\vec{\nabla}(X)\right\rangle$$

II. Oscil·lador Harmònic

$$H = \frac{P^2}{2m}+\frac{1}{2}m\omega^2X^2$$

Solució algebraica:

$$a\equiv\sqrt{\frac{m\omega}{2\hbar}}\left(X+\frac{i}{m\omega}P\right) \qquad X=\sqrt{\frac{\hbar}{2m\omega}}(a+a^\dagger) \quad P=-i\sqrt{\frac{\hbar}{2m\omega}}(a-a^\dagger) \qquad [a,a^\dagger]=\mathbb{I} \qquad N\equiv a^\dagger a$$

$$H=\hbar\omega\left(N+\frac{1}{2}\right) \qquad N|n\rangle = n|n\rangle \qquad a|n\rangle=\sqrt{n}|n-1\rangle \qquad a^\dagger|n\rangle=\sqrt{n+1}|n+1\rangle$$

$$E_n=\hbar\omega\left(n+\frac{1}{2}\right) \qquad H|n\rangle=\hbar\omega\left(n+\frac{1}{2}\right)|n\rangle \qquad |n\rangle=\frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle \qquad \langle n|X|n\rangle=0 \quad \langle n|P|n\rangle=0$$

Solució tradicional:

$$\psi_0(x)=\frac{1}{\pi^{\frac{1}{4}}x_0^{\frac{1}{2}}}\mathrm{e}^{-\frac{1}{2}\left(\frac{x}{x_0}\right)^2} \qquad \psi_1(x)=\sqrt{2}\frac{x}{x_0}\psi_0(x) \qquad \psi_n(x)=\sqrt{\frac{1}{x_0\sqrt{\pi}2^nn!}}H_n\left(\frac{x}{x_0}\right)\mathrm{e}^{-\frac{x^2}{2x_0^2}}$$

$$x_0=\sqrt{\frac{\hbar}{m\omega}}$$

Estats coherents:

$$a|\lambda\rangle=\lambda|\lambda\rangle \qquad |\lambda\rangle=\mathrm{e}^{-\frac{|\lambda|^2}{2}}\mathrm{e}^{\lambda a^\dagger}|0\rangle \qquad E_{|\lambda\rangle}=\hbar\omega\left(|\lambda|^2+\frac{1}{2}\right)$$

III. Simetries, Lleis de Conservació i Moment Angular

Translacions:

$$U(\vec{a})=\mathrm{e}^{-i\frac{\vec{a}\cdot\vec{P}}{\hbar}}, \quad U(\vec{a})\vec{X}U^\dagger(\vec{a})=\vec{X}-\vec{a}\mathbb{I}$$

Paritat:

$$U_p\vec{X}U_p^\dagger=-\vec{X}$$

Operador de Casimir: $A(J_i)$

$$[A,J_i]=0$$

Moment angular orbital:

$$\vec{L}=-i\hbar\vec{R}\times\vec{\nabla}$$

Moment angular general:

$$[J_i,J_j]=i\hbar\epsilon_{ijk}J_k, \quad [\vec{J}^2,J_i]=0$$

$$\vec{J}^2|j,m\rangle=\hbar^2j(j+1)|j,m\rangle, \quad J_z|j,m\rangle=m\hbar|j,m\rangle$$

$$J_+\equiv J_x+iJ_y=J_-^\dagger, \quad J_-\equiv J_x-iJ_y=J_+^\dagger$$

$$J_\pm|j,m\rangle=N_\pm|j,m\pm1\rangle, \; N_\pm=\hbar\sqrt{j(j+1)-m(m\pm1)}$$

Moment angular d'spin:

$$\text{En 2D: }\vec{S}=\frac{\hbar}{2}\vec{\sigma}$$

Rotacions:

$$U_{\hat{n}}=\mathrm{e}^{-i\frac{\alpha}{\hbar}\hat{n}\cdot\vec{J}}$$

IV. Mètodes Aproximats

Pertorbacions indep. temps sense deg.:

$$H = H_0 + \lambda H' \qquad |\psi_n\rangle = \sum_{j=0}^{\infty} \lambda^j \left|\psi_n^{(j)}\right\rangle \qquad E_n = \sum_{j=0}^{\infty} \lambda^j E_n^{(j)}$$

$$E_n^{(1)} = \left\langle \psi_n^{(0)} \middle| H' \middle| \psi_n^{(0)} \right\rangle \qquad E_n^{(2)} = \sum_{k \neq n} \frac{\left\langle \psi_k^{(0)} \middle| H' \middle| \psi_n^{(0)} \right\rangle \left\langle \psi_k^{(0)} \middle| H' \middle| \psi_n^{(0)} \right\rangle}{E_n^{(0)} - E_k^{(0)}}$$

$$\left|\psi_n^{(1)}\right\rangle = \sum_{k \neq n} a_{nk}^{(1)} \left|\psi_k^{(0)}\right\rangle \qquad a_{nk}^{(1)} = \frac{\left\langle \psi_k^{(0)} \middle| H' \middle| \psi_n^{(0)} \right\rangle}{E_k^{(0)} - E_n^{(0)}}$$

Mètode Variacional

$$H |\psi_n\rangle = E_n |\psi_n\rangle \quad |\phi\rangle = |\psi_n\rangle \qquad E_0 \leq E[\phi] = \frac{\langle \phi | H | \phi \rangle}{\langle \phi | \phi \rangle}$$