

# Ampliación de Mecánica Teórica

pod

Primavera, 2002

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\vec{q}}} - \frac{\partial L}{\partial \vec{q}} = 0, \quad L = T - V$$

$$\vec{p} = \frac{\partial L}{\partial \dot{\vec{q}}}, \quad H = \vec{p} \cdot \dot{\vec{q}} - L$$

$$\dot{\vec{q}} = \frac{\partial H}{\partial \vec{p}}, \quad \dot{\vec{p}} = -\frac{\partial H}{\partial \vec{q}}, \quad \frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + [f, H]$$

$$[f, g] = \frac{\partial f}{\partial \vec{q}} \frac{\partial g}{\partial \vec{p}} - \frac{\partial f}{\partial \vec{p}} \frac{\partial g}{\partial \vec{q}} = \left( \frac{\partial f}{\partial \vec{x}} \right)^t J \left( \frac{\partial g}{\partial \vec{x}} \right)$$

$$\dot{p}_i = [p_i, H], \quad \dot{q}_i = [q_i, H], \quad [p_i, q_j] = -\delta_{ij}$$

$$\vec{x} = (\vec{q}, \vec{p}), \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \dot{\vec{x}} = J \frac{\partial H}{\partial \vec{x}} = [\vec{x}, H]$$

Transformaciones canónicas.  $H' = H + \frac{\partial F}{\partial t}$

$$(\vec{p} \, d\vec{q} - H \, dt) - (\vec{P} \, d\vec{Q} - H' \, dt) = dF$$

1.  $F = F_1(q, Q, t)$ :

$$\vec{p} = \frac{\partial F_1}{\partial \vec{q}}, \quad \vec{P} = -\frac{\partial F_1}{\partial \vec{Q}}$$

2.  $F = F_2(q, P, t) - \vec{Q} \vec{P}$ :

$$\vec{p} = \frac{\partial F_2}{\partial \vec{q}}, \quad \vec{Q} = \frac{\partial F_2}{\partial \vec{P}}$$

3.  $F = \vec{q} \vec{p} + F_3(q, Q, t)$ :

$$\vec{q} = -\frac{\partial F_3}{\partial \vec{q}}, \quad \vec{P} = -\frac{\partial F_3}{\partial \vec{Q}}$$

4.  $F = \vec{q} \vec{p} - \vec{Q} \vec{P} + F_4(p, P, t)$ :

$$\vec{q} = -\frac{\partial F_4}{\partial \vec{p}}, \quad \vec{Q} = \frac{\partial F_4}{\partial \vec{P}}$$

$$\dot{\vec{X}} = M J M^t \frac{\partial H}{\partial \vec{X}} \rightarrow M J M^t = J$$

Invariante integral  $\frac{d}{dt} I = 0$

$$I(t) = \int_{\Omega_t} \dots \int d\vec{p} \, d\vec{q} \, \rho(\vec{q}, \vec{p}, t) \leftrightarrow \frac{\partial \rho}{\partial t} + [\rho, H] = 0$$

$$\frac{\partial S}{\partial t} + H \left( \vec{q}, \frac{\partial S}{\partial \vec{q}}, t \right) = 0, \quad \vec{p} = \frac{\partial S}{\partial \vec{q}}, \quad \vec{\beta} = \frac{\partial S}{\partial \vec{\alpha}}$$

$$J_k(\vec{\alpha}) = \oint p_k(q_k, \vec{\alpha}) \, dq_k, \quad \dot{\vec{\omega}} = \frac{\partial H'}{\partial \vec{J}} = \vec{v}(\vec{J})$$

Variación de las constantes  $H = H_0 + \lambda H_1$ , t.c.  $S_0$

$$\vec{\alpha}_1(t) = - \int dt \frac{\partial H_1}{\partial \vec{\beta}}$$

$$\vec{\alpha}_2(t) = - \int dt \left\{ \frac{\partial^2 H_1}{\partial \vec{\alpha} \partial \vec{\beta}} \cdot \vec{\alpha}_1 + \frac{\partial^2 H_1}{\partial \vec{\beta} \partial \vec{\beta}} \cdot \vec{\beta}_1 \right\}$$

$$\vec{\beta}_1(t) = \int dt \frac{\partial H_1}{\partial \vec{\alpha}}$$

$$\vec{\beta}_2(t) = \int dt \left\{ \frac{\partial^2 H_1}{\partial \vec{\beta} \partial \vec{\alpha}} \cdot \vec{\beta}_1 + \frac{\partial^2 H_1}{\partial \vec{\alpha} \partial \vec{\alpha}} \cdot \vec{\alpha}_1 \right\}$$

Met.media

$$H'(\alpha, \beta, t) \approx \langle H_1(\alpha, \beta, t) \rangle = \frac{1}{T_0} \int_{T_0} dt' H_1(\alpha, \beta, t')$$

Poincaré-Von Ziepel

$$E_1 = H_0^{(1)} = \int_{[0,1]^n} d\vec{\omega}_0 H_1(\vec{J}) \equiv \langle H_1(\vec{J}) \rangle$$

$$E_2 = \langle H_2(\vec{J}, \vec{\omega}_0) \rangle + \frac{1}{2} \left\langle \frac{\partial S_1}{\partial \vec{\omega}_0} \cdot \frac{\partial^2 H_0}{\partial \vec{J} \partial \vec{J}} \cdot \frac{\partial S_1}{\partial \vec{\omega}_0} \right\rangle + \left\langle \frac{\partial H_1}{\partial \vec{J}} \cdot \frac{\partial S_1}{\partial \vec{\omega}_0} \right\rangle$$

$$S_1(\vec{\omega}_0, \vec{J}) = \frac{-1}{2\pi i} \sum_{\vec{m} \neq 0} \frac{H_{\vec{m}}^{(1)}(\vec{J})}{\vec{m} \cdot \vec{\nu}_0(\vec{J})} e^{2\pi i \vec{m} \cdot \vec{\omega}_0}$$

$$\vec{\nu} = \vec{\nu}_0 + \epsilon \partial[E_1]\vec{J} + \epsilon^2 \partial[E_2]\vec{J} + \dots$$

$$J_0 = J + \epsilon \frac{\partial S_1}{\partial \vec{\omega}_0} + \dots$$

$$\vec{\omega} = \omega_0 + \epsilon \frac{\partial S_1}{\partial \vec{J}} + \dots$$

$$(n=1) E_1(J) = H_1(J, \omega_0) + \nu_0(\vec{J}) \frac{\partial S_1}{\partial \omega_0}$$

- Taylor  $E = H_0(\frac{\partial S}{\partial \vec{\omega}_0}) + \epsilon H_1(\frac{\partial S}{\partial \vec{\omega}_0}, \vec{\omega}_0) + \dots$
- Desenvol i  $E = H'_0 + \epsilon H'_1 + \dots$
- Fourier  $S_{\vec{m}}$
- Integrar  $\vec{\omega}_0$ .

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Twist Map

$$J_{n+1} = J_n \quad , \quad \omega_{n+1} = \alpha(J_{n+1}) + \omega_n$$

Pertorb.  $\epsilon H_1$

$$J_{n+1} = J_n + \epsilon f(J_{n+1}, \omega_n)$$

$$\omega_{n+1} = \alpha(J_{n+1}) + \omega_n + \epsilon g(J_{n+1}, \omega_n)$$

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Teor. adiabática

$$\lambda = \lambda(\epsilon t) \implies \Delta J = o(\epsilon^2)$$

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Sist. dinámicos  $\dot{\vec{x}} = F(\vec{x}, t)$

$$\vec{k} = \text{cte} \quad , \quad \vec{F} = \vec{F}(\vec{x}) \iff \vec{F} \cdot \overrightarrow{\text{grad}} = 0$$

$$I(t) = \int \dots \int_{\Omega(t)} d^N \vec{x} \rho(\vec{x}, t) = \text{cte} \iff$$

$$\iff \frac{\partial \rho}{\partial t} + \text{div} [\rho(\vec{x}, t) \vec{F}(\vec{x}, t)] = 0$$

Rectificat  $\{x_i\} \rightarrow \{y_i\}$  tq  $y_i = \delta_{1i}$

$$F_k(\vec{x}) = \sum_l \left. \frac{\partial F_k}{\partial x_l} \right|_{\vec{x}=\vec{0}} x_l + o(\vec{x}^2) = \sum_l a_{kl} x_l \quad , \quad \dot{\vec{x}} = A\vec{x}$$

$$\lambda_k = \alpha_k + i\omega_k \longrightarrow x_k = \sum_l c_{kl} e^{\alpha_l t} \cos(\omega_l t + \varphi_l)$$

- $\lambda_1 \neq \lambda_2 \in \mathbb{R} - \{0\}$ 
  - $\lambda_1/\lambda_2 > 0$  nodo, parábola ( $\lambda_i > 0$  inestable)
  - $\lambda_1/\lambda_2 < 0$  p. ensilladura, hipérbola
- $\lambda_1 = \lambda_2^*$  foco ( $\alpha > 0$  inestable,  $\alpha = 0$  centro)
- $\lambda_1 = \lambda_2$  nodo.

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Oscil. amónic (frec.  $\varpi$ )

$$q = \sqrt{\frac{J_0}{\pi \varpi m}} \sin 2\pi \omega_0 \quad , \quad p = \sqrt{\frac{J_0 \varpi m}{\pi}} \cos 2\pi \omega_0$$

$$F = \sum_{\vec{\ell}} a_{\vec{\ell}} e^{2\pi i \vec{\ell} \cdot \vec{\omega} t} \quad \text{on} \quad a_{\vec{\ell}} = e^{2\pi i \vec{\beta} \cdot \vec{\ell}} \int_{[0,1]^N} d^n \vec{\omega} F e^{-2\pi i \vec{\ell} \cdot \vec{\omega}}$$