On the Gravitational Field of a Mass Point according to Einstein's Theory †

by K. Schwarzschild

(Communicated January 13th, 1916 [see above p. 42].)

translation‡ and foreword by

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Foreword. This fundamental memoir contains the ORIGINAL form of the solution of Schwarzschild's problem. It is regular in the whole space-time, with the only exception of the origin of the spatial co-ordinates; consequently, it leaves no room for the science fiction of the black holes. (In the centuries of the decline of the Roman Empire people said: "Graecum est, non legitur"...).

§1. In his work on the motion of the perihelion of Mercury (see Sitzungsberichte of November 18th, 1915) Mr. Einstein has posed the following problem:

Let a point move according to the prescription:

$$\begin{cases} \delta \displaystyle\int ds = 0, \\ \quad where \\ ds = \sqrt{\Sigma g_{\mu\nu} dx_\mu dx_\nu} \quad \mu, \nu = 1, 2, 3, 4, \end{cases} \tag{1}$$

where the $g_{\mu\nu}$ stand for functions of the variables x, and in the variation the variables x must be kept fixed at the beginning and at the end of the path of integration. In short, the point shall move along a geodesic line in the manifold characterised by the line element ds.

The execution of the variation yields the equations of motion of the point:

$$\frac{d^2 x_\alpha}{ds^2} = \sum_{\mu,\nu} \Gamma^\alpha_{\mu\nu} \frac{dx_\mu}{ds}\frac{dx_\nu}{ds}, \quad \alpha, \beta = 1, 2, 3, 4, \tag{2}$$

where

$$\Gamma^\alpha_{\mu\nu} = -\frac{1}{2} \sum_\beta g^{\alpha\beta} \left(\frac{\partial g_{\mu\beta}}{\partial x_\nu} + \frac{\partial g_{\nu\beta}}{\partial x_\mu} - \frac{\partial g_{\mu\nu}}{\partial x_\beta} \right), \tag{3}$$

and the $g^{\alpha\beta}$ stand for the normalised minors associated to $g_{\alpha\beta}$ in the determinant $|g_{\mu\nu}|$.

According to Einstein's theory, this is the motion of a massless point in the gravitational field of a mass at the point $x_1 = x_2 = x_3 = 0$, if the "components of the gravitational field" Γ fulfil everywhere, with the exception of the point $x_1 = x_2 = x_3 = 0$, the "field equations"

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$$\sum_{\alpha} \frac{\partial \Gamma^{\alpha}_{\mu\nu}}{\partial x_{\alpha}} + \sum_{\alpha\beta} \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} = 0, \tag{4}$$

and if also the "equation of the determinant"

$$|g_{\mu\nu}| = -1 \tag{5}$$

is satisfied.

The field equations together with the equation of the determinant have the fundamental property that they preserve their form under the substitution of other arbitrary variables in lieu of x_1, x_2, x_3, x_4, as long as the determinant of the substitution is equal to 1.

Let x_1, x_2, x_3 stand for rectangular co-ordinates, x_4 for the time; furthermore, the mass at the origin shall not change with time, and the motion at infinity shall be rectilinear and uniform. Then, according to Mr. Einstein's list, *loc. cit.* p. 833, the following conditions must be fulfilled too:

1. All the components are independent of the time x_4.
2. The equations $g_{\rho 4} = g_{4\rho} = 0$ hold exactly for $\rho = 1, 2, 3$.
3. The solution is spatially symmetric with respect to the origin of the co-ordinate system in the sense that one finds again the same solution when x_1, x_2, x_3 are subjected to an orthogonal transformation (rotation).
4. The $g_{\mu\nu}$ vanish at infinity, with the exception of the following four limits different from zero:

$$g_{44} = 1, \quad g_{11} = g_{22} = g_{33} = -1.$$

The problem is to find out a line element with coefficients such that the field equations, the equation of the determinant and these four requirements are satisfied.

§2. Mr. Einstein showed that this problem, in first approximation, leads to Newton's law and that the second approximation correctly reproduces the known anomaly in the motion of the perihelion of Mercury. The following calculation yields the exact solution of the problem. It is always pleasant to avail of exact solutions of simple form. More importantly, the calculation proves also the uniqueness of the solution, about which Mr. Einstein's treatment still left doubt, and which could have been proved only with great difficulty, in the way shown below, through such an approximation method. The following lines therefore let Mr. Einstein's result shine with increased clearness.

§3. If one calls t the time, x, y, z, the rectangular co-ordinates, the most general line element that satisfies the conditions 1-3 is clearly the following:

$$ds^2 = Fdt^2 - G(dx^2 + dy^2 + dz^2) - H(xdx + ydy + zdz)^2$$

where F, G, H are functions of $r = \sqrt{x^2 + y^2 + z^2}$.

The condition (4) requires: for $r = \infty : F = G = 1, H = 0$.

When one goes over to polar co-ordinates according to $x = r\sin\vartheta\cos\phi$, $y = r\sin\vartheta\sin\phi$, $z = r\cos\vartheta$, the same line element reads:

$$\begin{aligned} ds^2 &= Fdt^2 - G(dr^2 + r^2 d\vartheta^2 + r^2 sin^2\vartheta d\phi^2) - Hr^2dr^2 \\ &= Fdt^2 - (G + Hr^2)dr^2 - Gr^2(d\vartheta^2 + sin^2\vartheta d\phi^2). \end{aligned} \tag{6}$$

Now the volume element in polar co-ordinates is equal to $r^2 \sin\vartheta dr d\vartheta d\phi$, the functional determinant $r^2\sin\vartheta$ of the old with respect to the new coordinates is different from 1; then the field equations would not remain in unaltered form if one would calculate with these polar co-ordinates, and one would have to perform a cumbersome transformation. However there is an easy trick to circumvent this difficulty. One puts:

$$x_1 = \frac{r^3}{3}, \quad x_2 = -\cos\vartheta, \quad x_3 = \phi. \tag{7}$$

Then we have for the volume element: $r^2 dr \sin\vartheta d\vartheta d\phi = dx_1 dx_2 dx_3$. The new variables are then *polar co-ordinates with the determinant 1.* They have the evident advantages of polar co-ordinates for the treatment of the problem, and at the same time, when one includes also $t = x_4$, the field equations and the determinant equation remain in unaltered form.

In the new polar co-ordinates the line element reads:

$$ds^2 = F dx_4^2 - \left(\frac{G}{r^4} + \frac{H}{r^2}\right) dx_1^2 - G r^2 \left[\frac{dx_2^2}{1 - x_2^2} + dx_3^2 (1 - x_2^2)\right], \tag{8}$$

for which we write:

$$ds^2 = f_4 dx_4^2 - f_1 dx_1^2 - f_2 \frac{dx_2^2}{1 - x_2^2} - f_3 dx_3^2 (1 - x_2^2). \tag{9}$$

Then f_1, $f_2 = f_3$, f_4 are three functions of x_1 which have to fulfil the following conditions:

1. For $x_1 = \infty$: $f_1 = 1/r^4 = (3x_1)^{-4/3}$, $f_2 = f_3 = r^2 = (3x_1)^{2/3}$, $f_4 = 1$.
2. The equation of the determinant: $f_1 \cdot f_2 \cdot f_3 \cdot f_4 = 1$.
3. The field equations.
4. Continuity of the f, except for $x_1 = 0$.

§4. In order to formulate the field equations one must first form the components of the gravitational field corresponding to the line element (9). This happens in the simplest way when one builds the differential equations of the geodesic line by direct execution of the variation, and reads out the components from these. The differential equations of the geodesic line for the line element (9) result from the variation immediately in the form:

$$0 = f_1 \frac{d^2 x_1}{ds^2} + \frac{1}{2}\frac{\partial f_4}{\partial x_1}\left(\frac{dx_4}{ds}\right)^2 + \frac{1}{2}\frac{\partial f_1}{\partial x_1}\left(\frac{dx_1}{ds}\right)^2 - \frac{1}{2}\frac{\partial f_2}{\partial x_1}\left[\frac{1}{1 - x_2^2}\left(\frac{dx_2}{ds}\right)^2 + (1 - x_2^2)\left(\frac{dx_3}{ds}\right)^2\right]$$

$$0 = \frac{f_2}{1 - x_2^2}\frac{d^2 x_2}{ds^2} + \frac{\partial f_2}{\partial x_1}\frac{1}{1 - x_2^2}\frac{dx_1}{ds}\frac{dx_2}{ds} + \frac{f_2 x_2}{(1 - x_2^2)^2}\left(\frac{dx_2}{ds}\right)^2 + f_2 x_2 \left(\frac{dx_3}{ds}\right)^2$$

$$0 = f_2 (1 - x_2^2)\frac{d^2 x_3}{ds^2} + \frac{\partial f_2}{\partial x_1}(1 - x_2^2)\frac{dx_1}{ds}\frac{dx_3}{ds} - 2 f_2 x_2 \frac{dx_2}{ds}\frac{dx_3}{ds}$$

$$0 = f_4 \frac{d^2 x_4}{ds^2} + \frac{\partial f_4}{\partial x_1}\frac{dx_1}{ds}\frac{dx_4}{ds}.$$

The comparison with (2) gives the components of the gravitational field:

$$\Gamma^1_{11} = -\frac{1}{2}\frac{1}{f_1}\frac{\partial f_1}{\partial x_1}, \quad \Gamma^1_{22} = +\frac{1}{2}\frac{1}{f_1}\frac{\partial f_2}{\partial x_1}\frac{1}{1-x_2^2},$$

$$\Gamma^1_{33} = +\frac{1}{2}\frac{1}{f_1}\frac{\partial f_2}{\partial x_1}(1-x_2^2),$$

$$\Gamma^1_{44} = -\frac{1}{2}\frac{1}{f_1}\frac{\partial f_4}{\partial x_1},$$

$$\Gamma^2_{21} = -\frac{1}{2}\frac{1}{f_2}\frac{\partial f_2}{\partial x_1}, \quad \Gamma^2_{22} = -\frac{x_2}{1-x_2^2}, \quad \Gamma^2_{33} = -x_2(1-x_2^2),$$

$$\Gamma^3_{31} = -\frac{1}{2}\frac{1}{f_2}\frac{\partial f_2}{\partial x_1}, \quad \Gamma^3_{32} = +\frac{x_2}{1-x_2^2},$$

$$\Gamma^4_{41} = -\frac{1}{2}\frac{1}{f_4}\frac{\partial f_4}{\partial x_1}$$

(the other ones are zero).

Due to the rotational symmetry around the origin it is sufficient to write the field equations only for the equator ($x_2 = 0$); therefore, since they will be differentiated only once, in the previous expressions it is possible to set everywhere since the beginning $1 - x_2^2$ equal 1. The calculation of the field equations then gives

$$a)\ \frac{\partial}{\partial x_1}\left(\frac{1}{f_1}\frac{\partial f_1}{\partial x_1}\right) = \frac{1}{2}\left(\frac{1}{f_1}\frac{\partial f_1}{\partial x_1}\right)^2 + \left(\frac{1}{f_2}\frac{\partial f_2}{\partial x_1}\right)^2 + \frac{1}{2}\left(\frac{1}{f_4}\frac{\partial f_4}{\partial x_1}\right)^2,$$

$$b)\ \frac{\partial}{\partial x_1}\left(\frac{1}{f_1}\frac{\partial f_2}{\partial x_1}\right) = 2 + \frac{1}{f_1 f_2}\left(\frac{\partial f_2}{\partial x_1}\right)^2,$$

$$c)\ \frac{\partial}{\partial x_1}\left(\frac{1}{f_1}\frac{\partial f_4}{\partial x_1}\right) = \frac{1}{f_1 f_4}\left(\frac{\partial f_4}{\partial x_1}\right)^2.$$

Besides these three equations the functions f_1, f_2, f_3 must fulfil also the equation of the determinant

$$d)\ f_1 f_2^2 f_4 = 1,\ i.\ e.\ \ \frac{1}{f_1}\frac{\partial f_1}{\partial x_1} + \frac{2}{f_2}\frac{\partial f_2}{\partial x_1} + \frac{1}{f_4}\frac{\partial f_4}{\partial x_1} = 0.$$

For now I neglect (b) and determine the three functions f_1, f_2, f_4 from (a), (c), and (d). (c) can be transposed into the form

$$c')\ \frac{\partial}{\partial x_1}\left(\frac{1}{f_4}\frac{\partial f_4}{\partial x_1}\right) = \frac{1}{f_1 f_4}\frac{\partial f_1}{\partial x_1}\frac{\partial f_4}{\partial x_1}.$$

This can be directly integrated and gives

$$c'')\ = \frac{1}{f_4}\frac{\partial f_4}{\partial x_1} = \alpha f_1, \quad (\alpha\ integration\ constant)$$

The addition of (a) and (c') gives

$$\frac{\partial}{\partial x_1}\left(\frac{1}{f_1}\frac{\partial f_1}{\partial x_1} + \frac{1}{f_4}\frac{\partial f_4}{\partial x_1}\right) = \left(\frac{1}{f_2}\frac{\partial f_2}{\partial x_1}\right)^2 + \frac{1}{2}\left(\frac{1}{f_1}\frac{\partial f_1}{\partial x_1} + \frac{1}{f_4}\frac{\partial f_4}{\partial x_1}\right)^2.$$

By taking (d) into account it follows

$$-2\frac{\partial}{\partial x_1}\left(\frac{1}{f_2}\frac{\partial f_2}{\partial x_1}\right) = 3\left(\frac{1}{f_2}\frac{\partial f_2}{\partial x_1}\right)^2.$$

By integrating

$$\frac{1}{\frac{1}{f_2}\frac{\partial f_2}{\partial x_1}} = \frac{3}{2}x_1 + \frac{\rho}{2} \quad (\rho \ \textit{integration constant})$$

or

$$\frac{1}{f_2}\frac{\partial f_2}{\partial x_1} = \frac{2}{3x_1+\rho}.$$

By integrating once more,

$$f_2 = \lambda(3x_1+\rho)^{2/3}. \quad (\lambda \ \textit{integration constant})$$

The condition at infinity requires: $\lambda = 1$. Then

$$f_2 = (3x_1+\rho)^{2/3}. \tag{10}$$

Hence it results further from (c") and (d)

$$\frac{\partial f_4}{\partial x_1} = \alpha f_1 f_4 = \frac{\alpha}{f_2^2} = \frac{\alpha}{(3x_1+\rho)^{4/3}}.$$

By integrating while taking into account the condition at infinity

$$f_4 = 1 - \alpha(3x_1+\rho)^{-1/3}. \tag{11}$$

Hence from (d)

$$f_1 = \frac{(3x_1+\rho)^{-4/3}}{1-\alpha(3x_1+\rho)^{-1/3}}. \tag{12}$$

As can be easily verified, the equation (b) is automatically fulfilled by the expressions that we found for f_1 and f_2.

Therefore all the conditions are satisfied apart from the *condition of continuity.* f_1 will be discontinuous when

$$1 = \alpha(3x_1+\rho)^{-1/3}, \quad 3x_1 = \alpha^3 - \rho.$$

In order that this discontinuity coincides with the origin, it must be

$$\rho = \alpha^3. \tag{13}$$

Therefore the condition of continuity relates in this way the two integration constants ρ and α.

The complete solution of our problem reads now:

$$f_1 = \frac{1}{R^4}\frac{1}{1-\alpha/R}, \quad f_2 = f_3 = R^2, \quad f_4 = 1-\alpha/R,$$

where the auxiliary quantity

$$R = (3x_1+\rho)^{1/3} = (r^3+\alpha^3)^{1/3}$$

has been introduced.

When one introduces these values of the functions f in the expression (9) of the line element and goes back to the usual polar co-ordinates one gets the line element that forms the exact solution of Einstein's problem:

$$ds^2 = (1-\alpha/R)dt^2 - \frac{dR^2}{1-\alpha/R} - R^2(d\vartheta^2 + \sin^2\vartheta d\phi^2), \quad R = (r^3+\alpha^3)^{1/3}. \tag{14}$$

The latter contains only the constant α that depends on the value of the mass at the origin.

§5. *The uniqueness of the solution* resulted spontaneously through the present calculation. From what follows we can see that it would have been difficult to ascertain the uniqueness from an approximation procedure in the manner of Mr. Einstein. Without the continuity condition it would have resulted:

$$f_1 = \frac{(3x_1+\rho)^{-4/3}}{1-\alpha(3x_1+\rho)^{-1/3}} = \frac{(r^3+\rho)^{-4/3}}{1-\alpha(r^3+\rho)^{-1/3}}.$$

When α and ρ are small, the series expansion up to quantities of second order gives:

$$f_1 = \frac{1}{r^4}\left[1+\frac{\alpha}{r} - 4/3\frac{\rho}{r^3}\right].$$

This expression, together with the corresponding expansions of f_2, f_3, f_4, satisfies up to the same accuracy all the conditions of the problem. Within this approximation the condition of continuity does not introduce anything new, since discontinuities occur spontaneously only in the origin. Then the two constants α and ρ appear to remain arbitrary, hence the problem would be physically undetermined. The exact solution teaches that in reality, by extending the approximations, the discontinuity does not occur at the origin, but at $r = (\alpha^3-\rho)^{1/3}$, and that one must set just $\rho = \alpha^3$ for the discontinuity to go in the origin. With the approximation in powers of α and ρ one should survey very closely the law of the coefficients in order to recognise the necessity of this link between α and ρ.

§6. Finally, one has still to derive the *motion of a point in the gravitational field*, the geodesic line corresponding to the line element (14). From the three facts, that the line element is homogeneous in the differentials and that its coefficients do not depend on t and on ϕ, with the variation we get immediately three intermediate integrals. If one also restricts himself to the motion in the equatorial plane ($\vartheta = 90^o$, $d\vartheta = 0$) these intermediate integrals read:

$$(1-\alpha/R)\left(\frac{dt}{ds}\right)^2 - \frac{1}{1-\alpha/R}\left(\frac{dR}{ds}\right)^2 - R^2\left(\frac{d\phi}{ds}\right)^2 = const. = h, \tag{15}$$

$$R^2\frac{d\phi}{ds} = const. = c, \tag{16}$$

$$(1-\alpha/R)\frac{dt}{ds} = const. = 1 \quad (\textit{determination of the time unit}). \tag{17}$$

From here it follows

$$\left(\frac{dR}{d\phi}\right)^2 + R^2(1-\alpha/R) = \frac{R^4}{c^2}[1-h(1-\alpha/R)]$$

or with $1/R = x$

$$\left(\frac{dx}{d\phi}\right)^2 = \frac{1-h}{c^2} + \frac{h\alpha}{c^2}x - x^2 + \alpha x^3. \tag{18}$$

If one introduces the notations: $c^2/h = B$, $(1-h)/h = 2A$, this is identical to Mr. Einstein's equation (11), *loc. cit.* and gives the observed anomaly of the perihelion of Mercury.

Actually Mr. Einstein's approximation for the orbit goes into the exact solution when one substitutes for r the quantity

$$R = (r^3 + \alpha^3)^{1/3} = r\left(1 + \frac{\alpha^3}{r^3}\right)^{1/3}.$$

Since α/r is nearly equal to twice the square of the velocity of the planet (with the velocity of light as unit), for Mercury the parenthesis differs from 1 only for quantities of the order 10^{-12}. Therefore r is virtually identical to R and Mr. Einstein's approximation is adequate to the strongest requirements of the practice.

Finally, the exact form of the third Kepler's law for circular orbits will be derived. Owing to (16) and (17), when one sets $x = 1/R$, for the angular velocity $n = d\phi/dt$ it holds

$$n = cx^2(1 - \alpha x).$$

For circular orbits both $dx/d\phi$ and $d^2x/d\phi^2$ must vanish. Due to (18) this gives:

$$0 = \frac{1-h}{c^2} + \frac{h\alpha}{c^2}x - x^2 + \alpha x^3, \quad 0 = \frac{h\alpha}{c^2} - 2x + 3\alpha x^2.$$

The elimination of h from these two equations yields

$$\alpha = 2c^2x(1 - \alpha x)^2.$$

Hence it follows

$$n^2 = \frac{\alpha}{2}x^3 = \frac{\alpha}{2R^3} = \frac{\alpha}{2(r^3 + \alpha^3)}.$$

The deviation of this formula from the third Kepler's law is totally negligible down to the surface of the Sun. For an ideal mass point, however, it follows that the angular velocity does not, as with Newton's law, grow without limit when the radius of the orbit gets smaller and smaller, but it approaches a determined limit

$$n_0 = \frac{1}{\alpha\sqrt{2}}.$$

(For a point with the solar mass the limit frequency will be around 10^4 per second). This circumstance could be of interest, if analogous laws would rule the molecular forces.