Càlcul vectorial

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Operadors vectorials

1. Coordenades cartesianes

Siguin:

\[f(x,y,z) \quad \vec{f}(\vec{r}) = f_x(\vec{r}) \hat{i} + f_y(\vec{r}) \hat{j} + f_z(\vec{r}) \hat{k}\]

- Gradient:
  \[\vec{\nabla} \cdot \vec{f} = \frac{\partial f_x}{\partial x} \hat{i} + \frac{\partial f_y}{\partial y} \hat{j} + \frac{\partial f_z}{\partial z} \hat{k}\]

- Divergència:
  \[\vec{\nabla} \cdot \vec{f} = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}\]

- Rotacional:
  \[\vec{\nabla} \times \vec{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix}\]

- Laplacià:
  \[\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\]

2. Coordenades Polars

Siguin:

\[(x,y) \rightarrow (\rho,\theta); \quad \vec{f} = f_\rho \hat{r} + f_\theta \hat{\theta}\]

- Gradient:
  \[\vec{\nabla} \cdot \vec{f} = \frac{\partial f_\rho}{\partial \rho} \hat{r} + \frac{\partial f_\theta}{\partial \theta} \hat{\theta}\]

- Divergència:
  \[\vec{\nabla} \cdot \vec{f} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho f_\rho) + \frac{1}{\rho} \frac{\partial f_\theta}{\partial \theta} + \frac{1}{\rho} \frac{\partial f_\phi}{\partial \phi}\]

- Rotacional:
  \[\vec{\nabla} \times \vec{f} = \frac{1}{\rho^2} \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{\phi} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ f_\rho & f_\theta & f_\phi \end{vmatrix}\]

- Laplacià:
  \[\nabla^2 f = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} (\rho^2 \frac{\partial f}{\partial \rho}) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \phi^2}\]

3. Coordenades cilíndriques

\[f(x,y,z) \rightarrow f(\rho,\phi,z); \quad \vec{f}(\vec{r}) = (f_\rho, f_\phi, f_z)\]

- Gradient:
  \[\vec{\nabla} \cdot \vec{f} = \frac{\partial f_\rho}{\partial \rho} + \frac{1}{\rho} \frac{\partial f_\phi}{\partial \phi} + \frac{\partial f_z}{\partial z}\]

- Divergència:
  \[\vec{\nabla} \cdot \vec{f} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho f_\rho) + \frac{1}{\rho} \frac{\partial f_\phi}{\partial \phi} + \frac{\partial f_z}{\partial z}\]

- Rotacional:
  \[\vec{\nabla} \times \vec{f} = \frac{1}{\rho} \begin{vmatrix} \hat{\rho} & \hat{\phi} & \hat{z} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial \theta} \\ f_\rho & f_\phi & f_z \end{vmatrix}\]

- Laplacià:
  \[\nabla^2 f = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho^2 \frac{\partial f}{\partial \rho}) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \phi^2}\]

4. Coordenades esfèriques

\[f(x,y,z) \rightarrow f(\rho,\theta,\phi); \quad \vec{f}(\vec{r}) = (f_\rho, f_\theta, f_\phi)\]

- Gradient:
  \[\vec{\nabla} \cdot \vec{f} = \frac{\partial f_\rho}{\partial \rho} + \frac{1}{\rho} \frac{\partial f_\theta}{\partial \theta} + \frac{1}{\rho \sin \theta} \frac{\partial f_\phi}{\partial \phi}\]

- Divergència:
  \[\vec{\nabla} \cdot \vec{f} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho^2 f_\rho) + \frac{1}{\rho \sin \theta} \frac{\partial f_\theta}{\partial \theta} + \frac{1}{\rho \sin \theta} \frac{\partial f_\phi}{\partial \phi}\]

- Rotacional:
  \[\vec{\nabla} \times \vec{f} = \frac{1}{\rho^2 \sin \theta} \begin{vmatrix} \hat{\rho} & \hat{\theta} & \hat{\phi} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ f_\rho & f_\theta & f_\phi \end{vmatrix}\]

- Laplacià:
  \[\nabla^2 f = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} (\rho^2 \frac{\partial f}{\partial \rho}) + \frac{1}{\rho^2 \sin \theta} \frac{\partial^2 f}{\partial \theta^2} + \frac{1}{\rho^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2}\]
CORBES I SUPERFÍCIES EN $\mathbb{R}^3$

Tenim $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$

- Equació d'una recta tangent a una corba en un punt $A(x_0, y_0, z_0)$:
  \[
  \frac{dx(t)}{dt}(x-x_0) + \frac{dy(t)}{dt}(y-y_0) + \frac{dz(t)}{dt}(z-z_0) = 0
  \]

- Equació del pla normal a la corba en $A \to \mathbf{r} - \mathbf{r}_0 \perp d\mathbf{r}/dt$:
  \[
  \frac{dx(t)}{dt}(x-x_0) + \frac{dy(t)}{dt}(y-y_0) + \frac{dz(t)}{dt}(z-z_0) = 0
  \]

Parametritzar en funció de la longitud d’arc:

- $\mathbf{r}(t) = (x(t), y(t), z(t)) = r(x(z), y(z), z)$

  - $\mathbf{r}'(t) = \mathbf{v}(t)$
  - $\mathbf{r}''(t)$

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Diferencials

- **Polars**
  - Posició: $d\mathbf{r} = d\rho \mathbf{\hat{\rho}} + \rho d\phi \mathbf{\hat{\phi}} + dz \mathbf{\hat{z}}$
  - Longitud: $ds^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2$
  - Superfície $dS = \rho d\rho d\phi dz$ (per $\rho =$cte.)
  - Volum $dV = \rho d\rho d\phi dz$

- **Cilíndrides**
  - Posició: $d\mathbf{r} = d\rho \mathbf{\hat{\rho}} + \rho d\phi \mathbf{\hat{\phi}} + dz \mathbf{\hat{z}}$
  - Longitud: $ds^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2$
  - Superfície $dS = \rho d\rho d\phi (per \rho =$cte.)
  - Volum $dV = \rho d\rho d\phi dz$

- **Esferiques**
  - Posició: $d\mathbf{r} = d\rho \mathbf{\hat{r}} + \rho d\theta \mathbf{\hat{\theta}} + \rho \sin \theta d\phi \mathbf{\hat{\phi}}$
  - Longitud: $ds^2 = d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2 \theta d\phi^2$
  - Superfície $dS = \rho^2 \sin \theta d\theta d\phi$ per a $r =$cte.
  - Volum $dV = \rho^2 \sin \theta d\rho d\theta d\phi$

Fórmules de Serret-Frenet: $\mathbf{\hat{s}}, \mathbf{\hat{t}}$ i $\mathbf{\hat{n}}$ formen un sistema de referència mòbil ortogonal, on

- $\mathbf{\hat{s}} = \frac{d\mathbf{r}}{ds}$; $\mathbf{\hat{n}} = \frac{1}{K} \frac{d\mathbf{\hat{t}}}{ds}$; $\mathbf{\hat{t}} = \frac{d\mathbf{\hat{r}}}{ds}$

En funció d’un paràmetre $t$ qualsevol:

- $\mathbf{\hat{s}}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$
- $\mathbf{\hat{b}}(t) = \frac{\mathbf{r}'(t) \times \mathbf{r}''(t)}{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}$
- $\mathbf{\hat{n}}(t) = \frac{(\mathbf{\hat{r}}(t) \times \mathbf{\hat{r}}''(t)) \times \mathbf{\hat{r}}(t)}{|(\mathbf{\hat{r}}(t) \times \mathbf{\hat{r}}''(t)) \times \mathbf{\hat{r}}(t)|}$