Formulario de Mecánica Teòrica

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I. PRINCIPIOS VARIACIONALES

Acción y Principio de Hamilton

$$I = \int_{t_1}^{t_2} L(q_i, \dot{q}_i; t)\mathrm{d}t \quad \delta I = \left(\frac{\mathrm{d}I}{\mathrm{d}\alpha}\right)_0$$

$$q_i(x,\alpha) = q_i(x,0) + \alpha\eta_i(x)$$

$$L(q_i, \dot{q}_i; t) = T - V \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

Definiciones

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \quad Q_j = \sum_i \mathbf{F}_i \frac{\partial \mathbf{r}_i}{\partial q_j} = -\frac{\partial V}{\partial q_j}$$

$$\frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_j} = \frac{\partial \mathbf{r}_i}{\partial q_j}$$

Teoremas de conservación

$$h(q_i, \dot{q}_i; t) = \sum_i \dot{q}_i p_i - L$$

$$T = T_0 + T_1 + T_2$$

Ecuaciones de Hamilton

$$H(q_i, p_i; t) = \sum_i \dot{q}_i p_i - L$$

$$\frac{\partial H}{\partial p_i} = \dot{q}_i \quad \frac{\partial H}{\partial q_i} = -\dot{p}_i$$

$$\frac{\partial H}{\partial t} = \frac{\partial h}{\partial t} = -\frac{\partial L}{\partial t} \qquad \frac{\partial H}{\partial t} = \frac{\mathrm{d}H}{\mathrm{d}t}$$

Principio de Mínima Acción

$$\Delta \int_{t_1}^{t_2} \sum_i p_i \dot{q}_i \mathrm{d}t = 0$$

II. TRANSF. CANÓNICAS

$$\sum_i p_i \dot{q}_i - H = \sum_i P_i \dot{Q}_i - K + \frac{\mathrm{d}F}{\mathrm{d}t}$$

Funciones generatriz

$$F = F_1(q_i, Q_i; t) \quad p_i = \frac{\partial F_1}{\partial q_i} \quad P_i = -\frac{\partial F_1}{\partial Q_i}$$

$$F = F_2(q_i, P_i; t) - Q_i P_i \quad p_i = \frac{\partial F_2}{\partial q_i} \quad Q_i = \frac{\partial F_2}{\partial P_i}$$

$$F = F_3(p_i, Q_i; t) + q_i p_i \quad q_i = -\frac{\partial F_3}{\partial p_i} \quad P_i = -\frac{\partial F_3}{\partial Q_i}$$

$$F = F_4(p_i, P_i; t) + q_i p_i - Q_i P_i \quad q_i = -\frac{\partial F_4}{\partial p_i} \quad Q_i = \frac{\partial F_4}{\partial P_i}$$

$$K(Q_i, P_i; t) = H(Q_i, P_i; t) + \frac{\partial F_i(Q_i, P_i; t)}{\partial t}$$

Condiciones directas

$$\left(\frac{\partial Q_i}{\partial q_j}\right)_{q,p} = \left(\frac{\partial p_j}{\partial P_i}\right)_{Q,P} \qquad \left(\frac{\partial Q_i}{\partial p_j}\right)_{q,p} = -\left(\frac{\partial q_j}{\partial P_i}\right)_{Q,P}$$

$$\left(\frac{\partial P_i}{\partial q_j}\right)_{q,p} = -\left(\frac{\partial p_j}{\partial Q_i}\right)_{Q,P} \qquad \left(\frac{\partial P_i}{\partial p_j}\right)_{q,p} = \left(\frac{\partial q_j}{\partial Q_i}\right)_{Q,P}$$

Notación simpléctica

$$\dot{\boldsymbol{\eta}} = \mathbf{J}\frac{\partial H}{\partial \boldsymbol{\eta}} \quad \dot{\boldsymbol{\zeta}} = \mathbf{M}\dot{\boldsymbol{\eta}} = \mathbf{MJ}\frac{\partial H}{\partial \boldsymbol{\eta}} = \mathbf{J}\frac{\partial H}{\partial \boldsymbol{\zeta}}$$

$$\mathbf{MJM}^{\mathrm{T}} = \mathbf{J}$$

Paréntesis de Poisson

$$[u,v] \equiv \sum_i \left(\frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}\right)$$

$$\frac{\mathrm{d}g}{\mathrm{d}t} = \frac{\partial g}{\partial t} + [g,H]_{p,q}$$

$$[q_j, q_k]_{q,p} = 0 = [p_j, p_k]_{q,p} \qquad [q_j, p_k]_{q,p} = \delta_{jk}$$

$$\dot{q}_i = [q_i, H] \qquad \dot{p}_i = [p_i, H]$$

$$[uv, w] = [u,w]v + u[v,w]$$

$$[u,[v,w]] + [v,[w,v]] + [w,[u,v]] = 0$$

III. EC. DE HAMILTON - JACOBI

Equación y relaciones

$$\dot{Q}_i = \frac{\partial K}{\partial P_i} = 0 \quad \dot{P}_i = -\frac{\partial K}{\partial Q_i} = 0$$

$$H(q_i, p_i; t) + \frac{\partial F}{\partial t} = 0$$

$F = F_2(q_i, P_i; t)$

$$H\left(q_1, \dots, q_n, \frac{\partial F_2}{\partial q_1}, \dots, \frac{\partial F_2}{\partial q_n}; t\right) + \frac{\partial F_2}{\partial t} = 0$$

$$P_i = \alpha_i \quad \to \quad F_2(q_i, P_i; t) = S(q_i, \alpha_i; t)$$

$$p_i = \frac{\partial S}{\partial q_i} = f(q_i, \alpha_i; t) \quad Q_i = \frac{\partial S}{\partial \alpha_i} = \beta_i$$

Transf. inversa:

$$q_i = q_i(Q, P; t) = q_i(\beta_k, \alpha_k; t)$$

$$p_i = f(q_i, \alpha_k; t) = p_i(\beta_k, \alpha_k; t)$$

Sistemas conservativos

$$S(q_i, \alpha_i; t) = W(q_i, \alpha_i) - \alpha t \qquad \alpha = E$$

$$H\left(q_1, \dots, q_n, \frac{\partial W}{\partial q_1}, \dots, \frac{\partial W}{\partial q_n}\right) = E$$

IV. SR NO INERCIALES

$$(\mathrm{d}\mathbf{r})_f = \mathrm{d}\boldsymbol{\theta} \times \mathbf{r} \ \mapsto \ \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_f = \frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}t} \times \mathbf{r} = \boldsymbol{\omega} \times \mathbf{r}$$

General (P se mueve):

$$\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_f = \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_r + \boldsymbol{\omega} \times \mathbf{r}$$

$$\left(\frac{\mathrm{d}\mathbf{r}'}{\mathrm{d}t}\right)_f = \left(\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t}\right)_f + \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_r + \boldsymbol{\omega} \times \mathbf{r}$$

$$\mathbf{v}_f = \mathbf{V} + \mathbf{v}_r + \boldsymbol{\omega} \times \mathbf{r}$$

$\mathbf{F} = m\mathbf{a}_f$. Si $\boldsymbol{\omega} = \text{const.}$

$$\mathbf{F} = m\ddot{\mathbf{R}} + m\mathbf{a}_r + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2m\boldsymbol{\omega} \times \mathbf{v}_r$$

$$\mathbf{a}_r = \left(\frac{\mathrm{d}\mathbf{v}_r}{\mathrm{d}t}\right)_r$$

SRNI:

$$\mathbf{F} = m\mathbf{a}_r = m\mathbf{a}_f - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m\boldsymbol{\omega} \times \mathbf{v}_r$$

$$\mathbf{F}_{ce} = -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

$$\mathbf{F}_{co} = -2m(\boldsymbol{\omega} \times \mathbf{v}_r)$$

V. SÓLIDO RÍGIDO

$$\mathbf{v}_r = 0 \ \mapsto \ \mathbf{v}_\alpha = \mathbf{V} + \boldsymbol{\omega} \times \mathbf{r}_\alpha$$

Tensor de inercia

$$I_{ij} = \sum_\alpha m_\alpha \left[\delta_{ij} \sum_k x_{\alpha,k}^2 - x_{\alpha,i} x_{\alpha,j}\right]$$

$$I_{ij} = \int \rho(\mathbf{r}) \left[\delta_{ij} \sum_k x_k^2 - x_i x_j\right] \mathrm{d}\mathbf{r}$$

$$T_{\text{rot}} = \frac{1}{2} \sum_\alpha m_\alpha (\boldsymbol{\omega} \times \mathbf{r}_\alpha)^2 = \frac{1}{2} \sum_{i,j} I_{ij} \omega_i \omega_j$$

Momento Angular

$$\mathbf{L} = \sum_\alpha m_\alpha [r_\alpha^2 \boldsymbol{\omega} - \boldsymbol{r}_\alpha (\mathbf{r}_\alpha \cdot \boldsymbol{\omega})] \qquad L_i = \sum_j I_{ij} \omega_j$$

$$\mathbf{L} = \mathcal{I} \cdot \boldsymbol{\omega} \qquad T_{\text{rot}} = \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{L}$$

Ejes principales

$$I_{ij} = I_i \delta_{ij} \qquad L_i = I_i \omega_i \qquad T_{\text{rot}} = \frac{1}{2} \sum_i I_i \omega_i^2$$

Teorema Steiner

$$I_{ij} = J_{ij} - M(a^2 \delta_{ij} - a_i a_j) \qquad X_i = a_i + x_i$$

Ángulos de Euler

$$\boldsymbol{\lambda} = \boldsymbol{\lambda}_\psi \boldsymbol{\lambda}_\theta \boldsymbol{\lambda}_\varphi$$

$$\omega_1 = \dot{\varphi} \sin\theta \sin\psi + \dot{\theta} \cos\psi$$
$$\omega_2 = \dot{\varphi} \sin\theta \cos\psi - \dot{\theta} \sin\psi$$
$$\omega_3 = \dot{\varphi} \cos\theta + \dot{\psi}$$

Ec. Euler para sólido rígido

i. Ausencia de fuerzas externas:

$$\psi \ \to \ \frac{\partial T}{\partial \psi} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\psi}}\right)$$

$$(I_i - I_j)\omega_i \omega_j - \sum_k I_k \dot{\omega}_k \epsilon_{ijk} = 0$$

ii. En presencia de fuerzas externas:

$$(I_i - I_j)\omega_i \omega_j - \sum_k (I_k \dot{\omega}_k - N_k) \epsilon_{ijk} = 0$$